RATIONALLY ISOTROPIC EXCEPTIONAL PROJECTIVE HOMOGENEOUS VARIETIES ARE LOCALLY ISOTROPIC

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Abstract. Assume that R is a local regular ring containing an infinite perfect field, or that R is the local ring of a point on a smooth scheme over an infinite field. Let K be the field of fractions of R and $char(K) \neq 2$. Let X be an exceptional projective homogeneous scheme over R. We prove that in most cases the condition $X(K) \neq \emptyset$ implies $X(R) \neq \emptyset$.

1. Introduction

We prove the following theorem.

Theorem 1. *Let R be local regular ring containing an infinite perfect field, or the local ring of a point on a smooth scheme over an infinite field, K be the fraction field of R and $char(K) \neq 2$. Let G be a split simple group of exceptional type (that is, E_6, E_7, E_8, F_4, or G_2), P be a parabolic subgroup of G, ξ be a class from $\mathrm{H}^1(R, G)$, and $X = {}_\xi(G/P)$ be the corresponding homogeneous space. Assume that $P \neq P_7$, P_8, $P_{7,8}$ in case $G = E_8$, $P \neq P_7$ in case $G = E_7$, and $P \neq P_1$ in case $G = E_7^{ad}$. Then the condition $X(K) \neq \emptyset$ implies $X(R) \neq \emptyset$.*

2. Purity of some H^1 functors

Let A be a commutative noetherian domain of finite Krull dimension with a fraction field F. We say that a functor $\mathcal{F}$ from the category of commutative A-algebras to the category of sets *satisfies purity* for A if we have

$$\mathrm{Im}\,[\mathcal{F}(A) \to \mathcal{F}(F)] = \bigcap_{\mathrm{ht}\,\mathfrak{p}=1} \mathrm{Im}\,[\mathcal{F}(A_{\mathfrak{p}}) \to \mathcal{F}(F)].$$

If $\mathcal{H}$ is an étale group sheaf we write $\mathrm{H}^i(-, \mathcal{H})$ for $\mathrm{H}^i_{\text{ét}}(-, \mathcal{H})$ below through the text. The following theorem is proven in the characteristic zero case [Pa2, Theorem 4.0.3]. We extend it here to an arbitrary characteristic case.

Theorem 2. *Let R be the regular local ring from Theorem 1 and $k \subset R$ be the subfield of R mentioned in that Theorem. Let*

$$(*) \qquad 1 \to Z \to G \to G' \to 1$$

be an exact sequence of algebraic k-groups, where G and G' are reductive and Z is a closed central subgroup scheme in G. If the functor $\mathrm{H}^1(-, G')$ satisfies purity for R then the functor $\mathrm{H}^1(-, G)$ satisfies purity for R as well.

Lemma 1. *Consider the category of R-algebras. The functor*

$$R' \mapsto \mathcal{F}(R') = \mathrm{H}^1_{fppf}(R', Z)/\,\mathrm{Im}\,(\delta_{R'}),$$

where δ is the connecting homomorphism associated to sequence (*), satisfies purity for R.

Proof. Similar to the proof of [Pa, Theorem 12.0.36] □

Lemma 2. *The map*

$$\mathrm{H}^2_{fppf}(R, Z) \to \mathrm{H}^2_{fppf}(K, Z)$$

is injective.

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Proof. Similar to the proof of [Pa, Theorem 13.0.38] □

Proof of Theorem 2. Reproduce the diagram chase from the proof of [Pa2, Theorem 4.0.3]. For that consider the commutative diagram

$$\begin{array}{ccccccccc}
\{1\} & \longrightarrow & \mathcal{F}(K) & \xrightarrow{\delta_K} & \mathrm{H}^1(K,G) & \xrightarrow{\pi_K} & \mathrm{H}^1(K,G') & \xrightarrow{\Delta_K} & \mathrm{H}^2_{fppf}(K,Z) \\
 & & \uparrow{\scriptstyle\alpha} & & \uparrow{\scriptstyle\beta} & & \uparrow{\scriptstyle\gamma} & & \uparrow{\scriptstyle\alpha_1} \\
\{1\} & \longrightarrow & \mathcal{F}(R) & \xrightarrow{\delta} & \mathrm{H}^1(R,G) & \xrightarrow{\pi} & \mathrm{H}^1(R,G') & \xrightarrow{\Delta} & \mathrm{H}^2_{fppf}(R,Z)
\end{array}$$

Let $\xi \in \mathrm{H}^1(K,G)$ be an R-unramified class and let $\bar{\xi} = \pi_K(\xi)$. Clearly, $\bar{\xi} \in \mathrm{H}^1(K,G')$ is R-unramified. Thus there exists an element $\bar{\xi}' \in \mathrm{H}^1(R,G')$ such that $\bar{\xi}'_K = \bar{\xi}$. The map α_1 is injective by Lemma 2. One has $\Delta(\bar{\xi}') = 0$, since $\Delta_K(\bar{\xi}) = 0$. Thus there exists $\xi' \in \mathrm{H}^1(R,G)$ such that $\pi(\xi') = \bar{\xi}'$.

The group $\mathcal{F}(K)$ acts on the set $\mathrm{H}^1(K,G)$, since Z is a central subgroup of the group G. If $a \in \mathcal{F}(K)$ and $\zeta \in \mathrm{H}^1(K,G)$, then we will write $a \cdot \zeta$ for the resulting element in $\mathrm{H}^1(K,G)$.

Further, for any two elements $\zeta_1, \zeta_2 \in \mathrm{H}^1(K,G)$, having the same images in $\mathrm{H}^1(K,G)$ there exists a unique element $a \in \mathcal{F}(K)$ such that $a \cdot \zeta_1 = \zeta_2$. These remarks hold for the cohomology of the ring R, and for the cohomology of the rings $R_{\mathfrak{p}}$, where $\mathfrak{p}$ runs over all height 1 prime ideals of R. Since the images of ξ'_K and ξ coincide in $\mathrm{H}^1(K,G')$, there exists a unique element $a \in \mathcal{F}(K)$ such that $a \cdot \xi'_K = \xi$ in $\mathrm{H}^1(K,G)$.

Lemma 3. *The above constructed element $a \in \mathcal{F}(K)$ is an R-unramified.*

Assuming Lemma 3 complete the proof of the Theorem. By Lemma 1 the functor $\mathcal{F}$ satisfies the purity for regular local rings containing the field k. Thus there exists an element $a' \in \mathcal{F}(R)$ with $a'_K = a$. It's clear that $\xi'' = a' \cdot \xi' \in \mathrm{H}^1(R,G)$ satisfies the equality $\xi''_K = \xi$. It remains to prove Lemma 3.

For that consider a height 1 prime ideal $\mathfrak{p}$ in R. Since ξ is an R-unramified there exists its lift up to an element $\tilde{\xi}$ in $\mathrm{H}^1(R_{\mathfrak{p}},G)$. Set $\xi'_{\mathfrak{p}}$ to be the image of the ξ' in $\mathrm{H}^1(R_{\mathfrak{p}},G)$. The classes $\pi_{\mathfrak{p}}(\tilde{\xi}), \pi_{\mathfrak{p}}(\xi'_{\mathfrak{p}}) \in \mathrm{H}^1(R_{\mathfrak{p}},G')$ being regarded in $\mathrm{H}^1(K,G')$ coincide.

The map

$$\mathrm{H}^1(R_{\mathfrak{p}},G') \to \mathrm{H}^1(K,G')$$

is injective by Lemma 4, formulated and proven below. Thus $\pi_{\mathfrak{p}}(\tilde{\xi}) = \pi_{\mathfrak{p}}(\xi'_{\mathfrak{p}})$. Therefore there exists a unique class $a_{\mathfrak{p}} \in \mathcal{F}(R_{\mathfrak{p}})$ such that $a_{\mathfrak{p}} \cdot \xi'_{\mathfrak{p}} = \tilde{\xi} \in \mathrm{H}^1(R_{\mathfrak{p}},G)$. So, $a_{\mathfrak{p},K} \cdot \xi'_K = \xi \in \mathrm{H}^1(K,G)$ and $a_{\mathfrak{p},K} \cdot \xi'_K = \xi = a \cdot \xi'_K$. Thus $a = a_{\mathfrak{p},K}$. Lemma 3 is proven.

□

To finish the proof of the theorem it remains to prove the following

Lemma 4. *Let H be a reductive group scheme over a discrete valuation ring A. Assume that for each A-algebra Ω with an algebraically closed field Ω the algebraic group H_Ω is connected. Let K be the fraction field of A. Then the map*

$$\mathrm{H}^1(R,H) \to \mathrm{H}^1(K,H)$$

is injective.

Proof. Let $\xi_0, \xi_1 \in \mathrm{H}^1(A,H)$. Let $\mathcal{H}_0$ be a principal homogeneous H-bundle representing the class ξ_0. Let H_0 be the inner form of the group scheme H, corresponding to $\mathcal{H}_0$. Let $X = Spec(A)$. For each X-scheme S there is a well-known bijection $\phi_S\colon \mathrm{H}^1(S,H) \to \mathrm{H}^1(S,H_0)$ of non-pointed sets. That bijection takes the principal homogeneous H-bundle $\mathcal{H}_0 \times_X S$ to the trivial principal homogeneous H_0-bundle $H_0 \times_X S$. That bijection respects to morphisms of X-schemes.

Assume that $\xi_{0,K} = \xi_{1,K}$. Then one has $* = \phi_K(\xi_{0,K}) = \phi_K(\xi_{1,K}) \in \mathrm{H}^1(K,H_0)$. The kernel of the map $\mathrm{H}^1(A,H_0) \to \mathrm{H}^1(K,H_0)$ is trivial by Nisnevich theorem [Ni]. Thus $\phi_A(\xi_1) = * =$

$\phi_A(\xi_0) \in \mathrm{H}^1(A, H_0)$. Whence $\xi_1 = \xi_0 \in \mathrm{H}^1(A, H)$. The Lemma is proven and the Theorem is proven as well.

□

Theorem 3. *The functor* $\mathrm{H}^1(-, \mathrm{PGL}_n)$ *satisfies purity.*

Proof. Let $\xi \in \mathrm{H}^1(-, \mathrm{PGL}_n)$ be an R-unramified element. Let $\delta : \mathrm{H}^1(-, \mathrm{PGL}_n) \to \mathrm{H}^2(-, G_m)$ be the boundary map corresponding to the short exact sequence of étale group sheaves

$$1 \to \mathbb{G}_m \to \mathrm{GL}_n \to \mathrm{PGL}_n \to 1.$$

Let D_ξ be a central simple K-algebra of degree n corresponding ξ. If $D_\xi \cong M_l(D')$ for a skew-field D', then there exists $\xi' \in \mathrm{H}^1(K, \mathrm{PGL}_{n'})$ such that $D' = D_{\xi'}$. Then $\delta(\xi') = [D'] = [D] = \delta(\xi)$. Replacing ξ by ξ', we may assume that $D := D_\xi$ is a central skew-field over K of degree n and the class $[D]$ is R-unramified.

Clearly, the class $\delta(\xi)$ is R-unramified. Thus there exists an Azumaya R-algebra A and an integer d such that $A_K = M_d(D)$.

There exists a projective left A-module P of finite rank such that each projective left A-module Q of finite rank is isomorphic to the left A-module P^m for an appropriative integer m (see [**?**, Cor.2]). In particular, two projective left A-modules of finite rank are isomorphic if they have the same rank as R-modules. One has an isomorphism $A \cong P^s$ of left A-modules for an integer s. Thus one has R-algebra isomorphisms $A \cong \mathrm{End}_A(P^s) \cong \mathrm{M}_s(\mathrm{End}_A(P))$. Set $B = \mathrm{End}_A(P)$. Observe, that $B_K = \mathrm{End}_{A_K}(P_K)$, since P is a finitely generated projective left A-module.

The class $[P_K]$ is a free generator of the group $K_0(A_K) = K_0(M_d(D)) \cong \mathbb{Z}$, since $[P]$ is a free generator of the group $K_0(A)$ and $K_0(A) = K_0(A_K)$. The P_K is a simple A_K-module, since $[P_K]$ is a free generator of $K_0(A_K)$. Thus $\mathrm{End}_{A_K}(P_K) = B_K$ is a skew-field.

We claim that the K-algebras B_K and D are isomorphic. In fact, $A_K = M_r(B_K)$ for an integer r, since P_K is a simple A_K-module. From the other side $A_K = M_d(D)$. As D, so B_K are skew-fields. Thus $r = d$ and D is isomorphic to B_K as K-algebras.

We claim further that B is an Azumaya R-algebra. That claim is local with respect to the étale topology on $\mathrm{Spec}(R)$. Thus it suffices to check the claim assuming that $\mathrm{Spec}(R)$ is sticktly henselian local ring. In that case $A = M_l(R)$ and $P = (R^l)^m$ as an $M_l(R)$-module. Thus $B = \mathrm{End}_A(P) = M_m(R)$, which proves the claim.

Since B_K is isomorphic to D, one has $m = n$. So, B is an Azumaya R-algebra, and the K-algebra B_K is isomorphic to D. Let $\zeta \in \mathrm{H}^1(R, \mathrm{PGL}_n)$ be class corresponding to B. Then $\zeta_K = \xi$, since $\delta(\zeta)_K = [B_K] = [D] = \delta(\xi) \in \mathrm{H}^2(K, \mathbb{G}_m)$.

□

Theorem 4. *The functor* $\mathrm{H}^1(-, \mathrm{Sim}_n)$ *satisfies purity.*

Proof. Let φ be a quadratic form over K whose similarity class represents $\xi \in \mathrm{H}^1(K, \mathrm{Sim}_n)$. Diagonalizing φ we may assume that $\varphi = \sum_{i=1}^n f_i \cdot t_i^2$ for certain non-zero elements $f_1, f_2, \dots, f_n \in K$. For each i write f_i in the form $f_i = \frac{g_i}{h_i}$ with $g_i, h_i \in R$ and $h_i \neq 0$.

There are only finitely many height one prime ideals $\mathfrak{q}$ in R such that there exists $0 \leq i \leq n$ with f_i not in $R_\mathfrak{q}$. Let $\mathfrak{q}_1, \mathfrak{q}_2, \dots, \mathfrak{q}_s$ be all height one prime ideals in R with that property and let $\mathfrak{q}_i \neq \mathfrak{q}_j$ for $i \neq j$.

For all other hight one prime ideals $\mathfrak{p}$ in R each f_i belongs to the group of units $R_\mathfrak{p}^\times$ of the ring $R_\mathfrak{p}$.

If $\mathfrak{p}$ is a hight one prime ideal of R which is not from the list $\mathfrak{q}_1, \mathfrak{q}_2, \dots, \mathfrak{q}_s$, then $\varphi = \sum_{i=1}^n f_i \cdot t_i^2$ may be regarded as a quadratic space over $R_\mathfrak{p}$. We will write ${}_\mathfrak{p}\varphi$ for that quadratic space over $R_\mathfrak{p}$. Clearly, one has $({}_\mathfrak{p}\varphi) \otimes_{R_\mathfrak{p}} K = \varphi$ as quadratic spaces over K.

For each $j \in \{1, 2, \dots, s\}$ choose and fix a quadratic space ${}_j\varphi$ over $R_{\mathfrak{q}_j}$ and a non-zero element $\lambda_j \in K$ such that the quadratic spaces $({}_j\varphi) \otimes_{R_{\mathfrak{q}_j}} K$ and $\lambda_j \cdot \varphi$ are isomorphic over K. The ring R is factorial since it is regular and local. Thus for each $j \in \{1, 2, \dots, s\}$ we may choose an element $\pi_j \in R$ such that firstly π_j generates the only maximal ideal in $R_{\mathfrak{q}_j}$ and secondly π_j is an invertible element in $R_\mathfrak{n}$ for each hight one prime ideal $\mathfrak{n}$ different from the ideal $\mathfrak{q}_j$.

Let $v_j\colon K^\times \to \mathbb{Z}$ be the discrete valuation of K corresponding to the prime ideal $\mathfrak{q}_j$. Set $\lambda = \prod_{i=1}^{s} \pi_j^{v_j(\lambda_j)}$ and

$$\varphi_{new} = \lambda \cdot \varphi.$$

Claim. The quadratic space φ_{new} is R-unramified. In fact, if a hight one prime ideal $\mathfrak{p}$ is different from each of $\mathfrak{q}_j$'s, then $v_\mathfrak{p}(\lambda) = 0$. Thus, $\lambda \in R_\mathfrak{p}^\times$. In that case $\lambda \cdot ({}_\mathfrak{p}\varphi)$ is a quadratic space over $R_\mathfrak{p}$ and moreover one have isomorphisms of quadratic spaces $(\lambda \cdot ({}_\mathfrak{p}\varphi)) \otimes_{R_\mathfrak{p}} K = \lambda \cdot \varphi = \varphi_{new}$. If we take one of $\mathfrak{q}_j$'s, then $\frac{\lambda}{\lambda_j} \in R_{\mathfrak{q}_j}^\times$. Thus, $\frac{\lambda}{\lambda_j} \cdot ({}_j\varphi)$ is a quadratic space over $R_{\mathfrak{q}_j}$. Moreover, one has

$$\frac{\lambda}{\lambda_j} \cdot ({}_j\varphi) \otimes_{R_\mathfrak{q}} K = \frac{\lambda}{\lambda_j} \cdot \lambda_j \cdot \varphi = \varphi_{new}.$$

The Claim is proven.

By [PP, Corollary 3.1] there exists a quadratic space $\tilde{\varphi}$ over R such that the quadratic spaces $\tilde{\varphi} \otimes_R K$ and φ_{new} are isomorphic over K. This shows that the similarity classes of the quadratic spaces $\tilde{\varphi} \otimes_R K$ and φ coincide. The theorem is proven. □

Theorem 5. *The functor* $\mathrm{H}^1(-, \mathrm{Sim}_n^+)$ *satisfies purity.*

Proof. Consider an element $\xi \in \mathrm{H}^1(K, \mathrm{Sim}_n^+)$ such that for any $\mathfrak{p}$ of height 1 ξ comes from $\xi_\mathfrak{p} \in \mathrm{H}^1(R_\mathfrak{p}, \mathrm{Sim}_n^+)$. Then the image of ξ in $\mathrm{H}^1(K, \mathrm{Sim}_n)$ by Theorem 4 comes from some $\zeta \in \mathrm{H}^1(R, \mathrm{Sim}_n)$. We have a short exact sequence

$$1 \to \mathrm{Sim}_n^+ \to \mathrm{Sim}_n \to \mu_2 \to 1,$$

and $R^\times/(R^\times)^2$ injects into $K^\times/(K^\times)^2$. Thus the element ζ comes actually from some $\zeta' \in \mathrm{H}^1(R, \mathrm{Sim}_n^+)$. It remains to show that the map

$$\mathrm{H}^1(K, \mathrm{Sim}_n^+) \to \mathrm{H}^1(K, \mathrm{Sim}_n)$$

is injective, or, by twisting, that the map

$$\mathrm{H}^1(K, \mathrm{Sim}^+(q)) \to \mathrm{H}^1(K, \mathrm{Sim}(q))$$

has the trivial kernel. The latter follows from the fact that the map

$$\mathrm{Sim}(q)(K) \to \mu_2(K)$$

is surjective (indeed, any reflection goes to $-1 \in \mu_2(K)$). □

3. Proof of the Main Theorem

Let ξ be a class from $\mathrm{H}^1(R, G)$, and $X = {}_\xi(G/P)$ be the corresponding homogeneous space. Denote by L a Levi subgroup of P.

Lemma 5. *Let L modulo its center be isomorphic to PGO_{2m}^+ (resp., PGO_{2m+1}^+ or $\mathrm{PGO}_{2m}^+ \times \mathrm{PGL}_2$). Denote by Ψ the closed subset in $\mathrm{X}^*(T)$ of type D_m (resp. B_m or $D_m + A_1$) corresponding to L, T stands for a maximal split torus in L. Assume that there is an element $\lambda \in \mathrm{X}^*(T)$ such that Ψ and λ generate a closed subset of type D_{m+1} (resp. B_{m+1} or $D_{m+1} + A_1$), and Ψ forms the standard subsystem of type D_m (resp. B_m or $D_m + A_1$) therein. Then there is a surjective map $L \to \mathrm{Sim}_{2m}^+$ (resp., $L \to \mathrm{Sim}_{2m+1}^+$ or $L \to \mathrm{Sim}_{2m}^+ \times \mathrm{PGL}_2$) whose kernel is a central closed subgroup scheme in L. In particular, the functor $\mathrm{H}^1(-, L)$ satisfies purity.*

Proof. It is easy to check that Sim_{2m}^+ (resp. Sim_{2m+1}^+) is a Levi subgroup in the split adjoint group of type D_{m+1} (resp. B_{m+1}). Now the first claim follows from [SGA, Exp. XXIII, Thm. 4.1], and the rest follows from Theorem 5 and Theorem 3. □

Lemma 6. *For any R-algebra S the map*

$$\mathrm{H}^1(S, L) \to \mathrm{H}^1(S, G)$$

is injective. Moreover, $X(S) \neq \emptyset$ if and only if ξ_S comes from $\mathrm{H}^1(S, L)$.

Proof. See [SGA, Exp. XXVI, Cor. 5.10]. □

Lemma 7. *Assume that the functor* $\mathrm{H}^1(-,L)$ *satisfies purity. Then* $X(K)\neq\emptyset$ *implies* $X(R)\neq\emptyset$.

Proof. By Lemma 6 ξ_K comes from some $\zeta\in\mathrm{H}^1(K,L)$, which is uniquely determined. Since X is smooth projective, for any prime ideal $\mathfrak{p}$ of height 1 we have $X(R_{\mathfrak{p}})\neq\emptyset$. By Lemma 6 $\xi_{R_{\mathfrak{p}}}$ comes from some $\zeta_{\mathfrak{p}}\in\mathrm{H}^1(R_{\mathfrak{p}},L)$. Now $(\zeta_{\mathfrak{p}})_K=\zeta$, and so by the purity assumption there is $\zeta'\in\mathrm{H}^1(R,L)$ such that $\zeta'_K=\zeta$.

Set ξ' to be the image of ζ' in $\mathrm{H}^1(R,G)$. We claim that $\xi'=\xi$. To prove this recall that by the construction $\xi'_K=\xi_K$. Further, there are natural in R-algebras bijections $\alpha_S:\mathrm{H}^1(S,G)\to\mathrm{H}^1(R,{}_{\xi'}G)$, which takes the ξ' to the distinguished element $*_R\in\mathrm{H}^1(R,{}_{\xi'}G)$. The R-group scheme ${}_{\xi'}G$ is isotropic and one has equalities

$$(\alpha_R(\xi))_K=\alpha_K(\xi_K)=\alpha_K(\xi'_K)=*'_K\in\mathrm{H}^1(K,{}_{\xi'}G).$$

Thus by [Pa, Theorem 1.0.1] one has

$$\alpha_R(\xi)=*'=\alpha_R(\xi')\in\mathrm{H}^1(R,{}_{\xi'}G).$$

The α_R is a bijection, whence

$$\xi=\xi'\in\mathrm{H}^1(R,G).$$

Lemma is proved. ∎

Lemma 8. *Let* $Q\leq P$ *be another parabolic subgroup,* $Y={}_{\xi}(G/Q)$. *Assume that* $X(K)\neq\emptyset$ *implies* $Y(K)\neq\emptyset$, *and* $Y(K)\neq\emptyset$ *implies* $Y(R)\neq\emptyset$. *Then* $X(K)\neq\emptyset$ *implies* $X(R)\neq\emptyset$.

Proof. Indeed, there is a map $Y\to X$, so $Y(R)\neq\emptyset$ implies $X(R)\neq\emptyset$. ∎

Proof of Theorem 1. By Lemma 8 we may assume that P corresponds to an item from the list of Tits [T, Table II]. We show case by case that $\mathrm{H}^1(-,L)$ satisfies purity, hence we are in the situation of Lemma 7.

If $P=B$ is the Borel subgroup, obviously $\mathrm{H}^1(-,L)=1$. In the case of index $E_{7,4}^{9}$ (resp. ${}^1E_{6,2}^{16}$) L modulo its center is isomorphic to $\mathrm{PGL}_2\times\mathrm{PGL}_2\times\mathrm{PGL}_2$ (resp. $\mathrm{PGL}_3\times\mathrm{PGL}_3$), and we may apply Theorem 2 and Theorem 3. In the all other cases we provide an element $\lambda\in\mathrm{X}^*(T)$ such that the assumption of Lemma 5 holds ($\tilde\alpha$ stands for the maximal root, enumeration follows [B]). The indices $E_{7,1}^{78}$, $E_{8,1}^{133}$ and $E_{8,2}^{78}$ are not in the list below since in those cases the L does not belong to one of the type D_m, B_m, $D_m\times A_1$. The index $E_{7,1}^{66}$ is not in the list below since in that case we need a weight λ which is not in the root lattice. So, the indices $E_{7,1}^{78}$, $E_{8,1}^{133}$, $E_{8,2}^{78}$ and $E_{7,1}^{66}$ are the exceptions in the statement of the Theorem.

Index	λ
${}^1E_{6,2}^{28}$	α_1
$E_{7,1}^{66}$	$-\omega_7$
$E_{7,1}^{48}$	$-\tilde\alpha$
$E_{7,2}^{31}$	α_1
$E_{7,3}^{28}$	α_1
$E_{8,1}^{91}$	$-\tilde\alpha$
$E_{8,2}^{66}$	α_8
$E_{8,4}^{28}$	α_1
$F_{4,1}^{21}$	$-\tilde\alpha$

∎

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